HCM HAUSDORFF CENTER FOR MATHEMATICS

An introduction to diagram algebras

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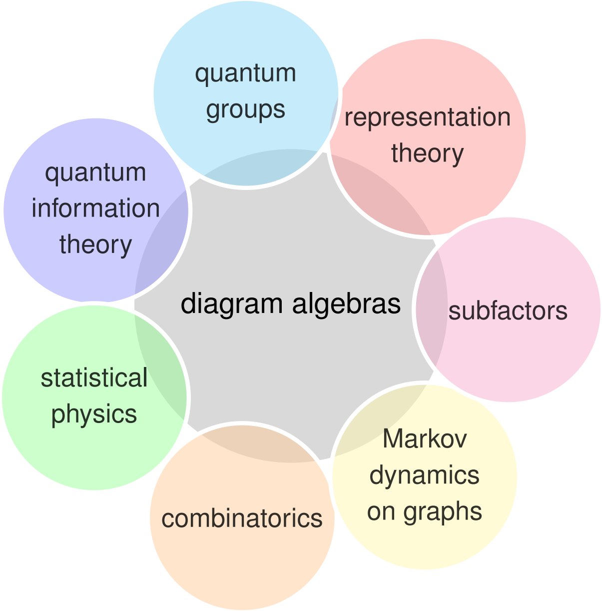
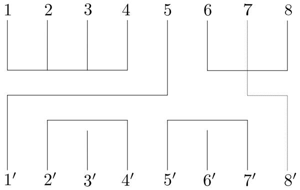

quantum groups
representation theory
quantum information theory
diagram algebras
subfactors
statistical physics
Markov dynamics on graphs
combinatorics

- Roughly speaking:
  - diagram algebra = complex vector space of formal linear combinations of diagrams.
  - multiplication = graphical operation on diagrams that is extended to linear combinations.

- In this talk, diagrams will typically be **partition diagrams** on k upper and k lower points.



- The diagram above depicts the partition of

$$\{\{1,2,3,4\},\{5,1'\},\{6,8\},\{7,8'\},\{2',4'\},\{3'\},\{5',7'\},\{6'\}\}.$$

- $\mathrm{Part}(k) = \{$partition diagrams on k upper and k lower points $\}, \; k \geq 0.$

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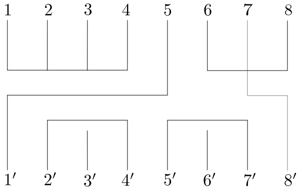
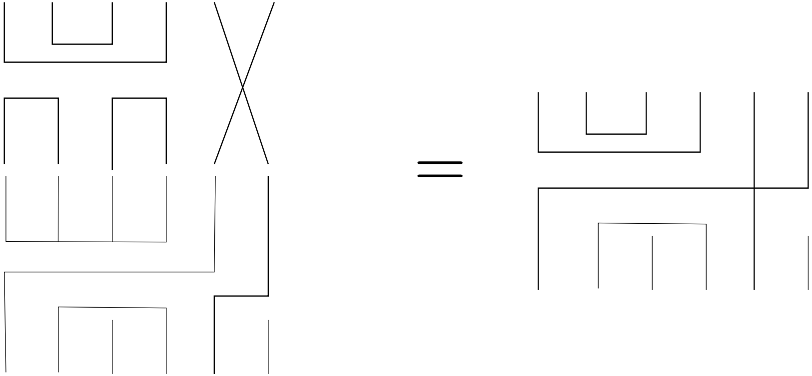


- The diagram above depicts the partition of

$$\{\{1,2,3,4\},\{5,1'\},\{6,8\},\{7,8'\},\{2',4'\},\{3'\},\{5',7'\},\{6'\}\}.$$

- $\mathrm{Part}(k) = \{$partition diagrams on k upper and k lower points $\},\ k \geq 0$.

- Partition diagrams can be multiplied by vertical concatenation and connecting lines.



Multiplication of two partitions $p, q \in \mathrm{Part}(6)$ yielding $p \cdot q \in \mathrm{Part}(6)$.

What is a diagram algebra?

Let $\mathcal{S} \subset \mathrm{Part}(k)$ be closed under multiplication of diagrams and let $\delta \in \mathbb{C}$.

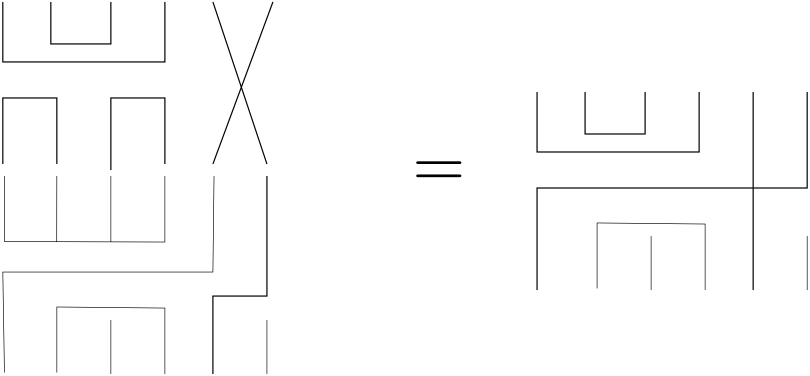
- k-th diagram algebra of $(\mathcal{S}, \delta)$:

$$A_{(\mathcal{S},\delta)}(k) = \left\{ \sum_{p \in \mathcal{S}} a_p \, e_p \; ; \; a_p \in \mathbb{C} \right\}$$

  = complex free vector spanned by basis $\{e_p \; ; \; p \in \mathcal{S}\}$.

- Multiplication:

$$e_p \cdot e_q = \delta^{\#\text{erased blocks in } p \cdot q} \, e_{p \cdot q}.$$



Here, we have erased one block, thus $e_p \cdot e_q = \delta\, e_{p \cdot q}$.

- $\mathcal{S} = \mathrm{Part}(k)$:

$$A_{(\mathcal{S},\delta)}(k) = \textbf{Partition algebras} \text{ (studied by Jones '94, Martin '96).}$$

- $\mathcal{S} = \{$diagr. s. t. every upper point is matched with exactly one lower point$\}$:

$$A_{(\mathcal{S},\delta)}(k) = \mathbb{C}[S_k].$$

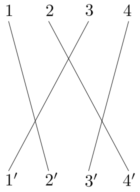


Diagram corresponding to the permutation (1243).

- $\mathcal{S} = \{$diagrams with blocks of size $2\} \subset \mathrm{Part}(k)$:

  $A_{(\mathcal{S},\delta)}(k) = \mathrm{Br}_\delta(k) =$ **Brauer algebras** (Brauer '37, Wenzl '88).

- $\mathcal{S} = \{$**noncrossing** diagrams with blocks of size $2\} \subset \mathrm{Part}(k)$:

  $A_{(\mathcal{S},\delta)}(k) = \mathrm{TL}_\delta(k) =$ **Temperley-Lieb algebras** (Jones '83).

- **Motzkin algebras** (nc. blocks of size one or two, Benkart-Halverson ’11);
- **Fuss-Catalan algebras** (nc. blocks of even size, Bisch-Jones ’95);
- **walled Brauer algebras** (Turaev ’89, Koike ’89, Nikitin ’07).

- The first examples of diagram algebras were introduced in order to describe *centralizers* of tensor product representations of compact groups.

Examples:

- Consider the standard representation

$$\pi : S_n \to L(\mathbb{C}^n)$$

of the symmetric group S_n on $\mathbb{C}^n$.
Then, the centralizer of the tensor product representation

$$\pi^{\otimes k} : S_n \to L((\mathbb{C}^n)^{\otimes k}),$$

is

$$\mathrm{End}_{S_n}((\mathbb{C}^n)^{\otimes k}) \cong \mathrm{Part}_{\delta=n}(k) \qquad \text{for} \quad 2k \leq n+1.$$

- The first examples of diagram algebras were introduced in order to describe *centralizers* of tensor product representations of compact groups.

Examples:

- Consider the standard representation

$$\pi : O_n \to L(\mathbb{C}^n)$$

of the orthogonal group O_n on $\mathbb{C}^n$.
Then, the centralizer of the tensor product representation

$$\pi^{\otimes k} : O_n \to L((\mathbb{C}^n)^{\otimes k}),$$

is

$$\mathrm{End}_{O_n}((\mathbb{C}^n)^{\otimes k}) \cong \mathrm{Br}_{\delta=n}(k) \qquad \text{for} \quad k < n.$$

Examples:

- Consider the standard representation of the special unitary group

$$SU(2) \to L(\mathbb{C}^2).$$

Then for the tensor product representation

$$\pi^{\otimes k} : SU(2) \to L((\mathbb{C}^2)^{\otimes k}),$$

we have

$$\mathrm{End}_{SU(2)}((\mathbb{C}^2)^{\otimes k}) \cong \mathrm{TL}_{\delta=2}(k) \qquad \text{for all} \quad k \geq 0.$$

- This remains true when $SU(2)$ is q-deformed to the quantum group $SU_q(2),\ q \in (0,1]$, when we choose $\delta = q + q^{-1}$.

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- This procedure can be 'inverted' (Banica-Speicher '09):

  for every diagram algebra $A_{(\mathcal{S},\delta)}(k)$ mentioned so far, one can construct a **easy/partition** quantum group G_n such that tensor products of its standard respresentation satisfy

  $$\mathrm{End}_{G_n}((\mathbb{C}^n)^{\otimes k}) \cong A_{(\mathcal{S},\delta=n)}(k)$$

  for a properly chosen range of k (relative to n).

diagram algebras	centralizer of
Brauer	O_n
Partition	$S_n, S'_n = \mathbb{Z}_2 \times S_n$
rook Brauer	$B_n, B'_n = \mathbb{Z}_2 \times B_n$
Orellana	H_n
walled Brauer	O_n^*
Temperley-Lieb $\mathrm{TL}_{\delta=n}(k)$	O_n^+
Temperley-Lieb $\mathrm{TL}_{\delta=\sqrt{n}}(2k)$	$S_n^+, S_n^{+'}$
Motzkin	$B_n^+, B_n^{+'}$
2-Fuss-Catalan	H_n^+
Weber	$B_n^{\#+}$

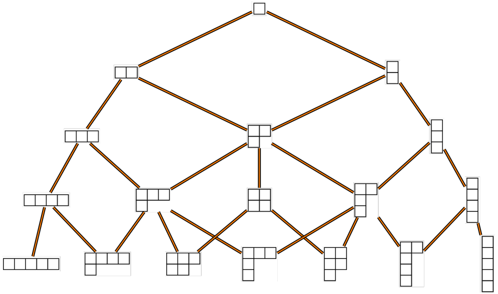
- Every diagram algebra $A_{(\mathcal{S},\delta)}(k)$ that arises from the Banica-Speicher framework of categories of partitions, is semisimple for **all but finitely many** values of $\delta \in \mathbb{C}$.

- What are the exact exceptional values for δ?
  - For $\mathrm{TL}_\delta(k)$: $\{2\cos(j\pi/k), 0 \leq j \leq k\}$ (Jones, Goodman-Wenzl '02)
  - For centralizers of easy groups (e.g. Brauer or partition algebra):

$$\{\text{exceptional}\} \subset \mathbb{Z}$$

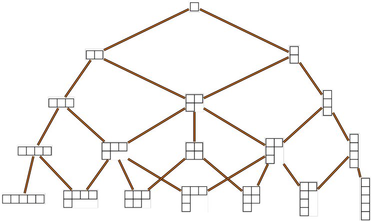
(see e.g Flake-Maaßen '20).

- The **Young graph** encodes a lot of information on the representation theory of the symmetric groups S_n.



The Young graph $\mathbb{Y}$: the branching graph of $S_1 \subset S_2 \subset S_3 \subset \ldots$

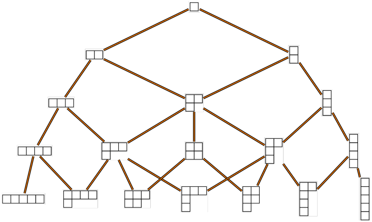
A return to the symmetric groups



- The nodes (Young diagrams) on the n-level enumerate the irreducible representations of S_n.
- Edges encode decomposition under restriction to S_{n-1}.
- **Example**: As representations of S_2, we have

$$\Pi_{(2,1)} \cong \Pi_{(2)} \oplus \Pi_{(1,1)}.$$

A return to the symmetric groups



- The Young graph also encodes the dimensions of the reps:

$$\dim(\Phi_D) = \text{ number of paths from root to } D.$$

- **Example**:

$$\dim\left(\Pi_{\yng(3,1)}\right) = 3.$$

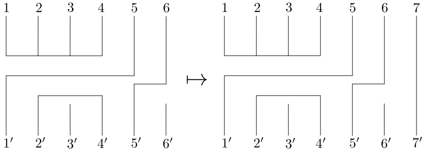
The diagram algebras, we have seen so far depended on

- a positive integer $k \geq 0$, the number of upper and lower points of the diagrams;
- the type of diagrams we allowed, e.g. we had $\mathcal{S} = \{$ diagrams with blocks of size two$\}$;
- a loop parameter $\delta \in \mathbb{C}$ which we now assume to be generic.

- If we fix $(\mathcal{S},\delta)$, we get a whole tower of diagram algebras

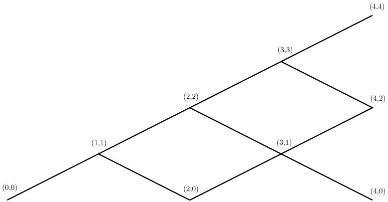
$$A_{(\mathcal{S},\delta)}(0) \subset A_{(\mathcal{S},\delta)}(1) \subset A_{(\mathcal{S},\delta)}(2) \subset \dots$$

where, on the level of partitions, we embed by adding a string to the right, e.g.

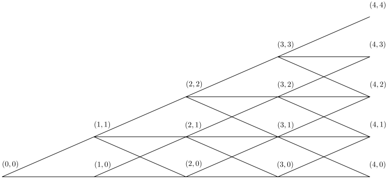


- We can derive the associated **branching graph** / **Bratteli diagram** by computing the representation theory of the algebras.

- Branching graph of $\cdots \subset \mathrm{TL}_\delta(k) \subset \ldots$ (noncrossing pairs) = *semi-Pascal graph* (Jones):

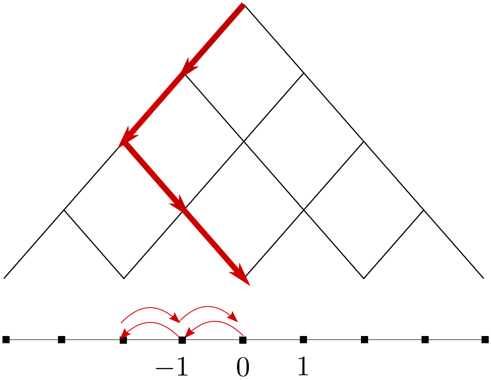


- Branching graph of $\cdots \subset \mathrm{Mo}_\delta(k) \subset \ldots$ (nc. pairs and singletons, Halverson-Benkart):



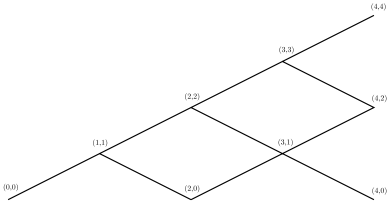
- For other diagram algebras, the branching graphs become increasingly inconvenient to draw.
- Luckily, there is a more condensed way of describing them:
- they all arise by a process dubbed **pascalization** (Vershik, Nikitin) from smaller graphs, their **principal graphs** (Jones).

- To describe this process, let us have a look at the *Pascal graph* $\mathcal{P}$.



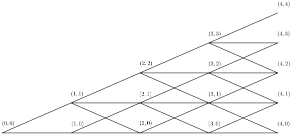
- Paths on Pascal graphs are trajectories of a walker on $\mathbb{Z}$ starting at 0.
- $\mathcal{P}$ is the pascalization of $\mathbb{Z}$, i.e. $\mathcal{P} = \mathcal{P}(\mathbb{Z})$.

- **semi-Pascal graph** $(\ldots \subset \mathrm{TL}_\delta(k) \subset \ldots)$:

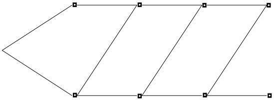


  - $s\mathcal{P} = \mathcal{P}(\mathbb{N})$.
  - Alternative interpretation as *Ballot paths* on $\mathbb{N} \times \mathbb{N}$ (useful for path counting).

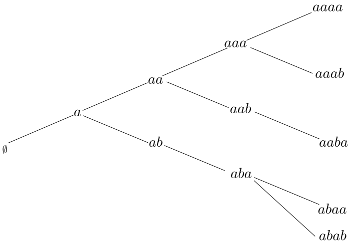
- **Motzkin graph** ($\cdots \subset \mathrm{Mo}_\delta(k) \subset \dots$):

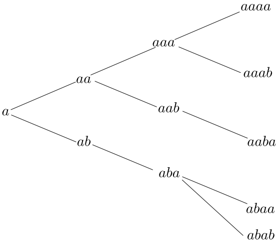


- Pascalization of the **ladder**:



- Alternative interpretations as
  - Lazy walks on the half-line $\mathbb{N}$;
  - *Motzkin paths* on $\mathbb{N} \times \mathbb{N}$ (useful for path counting).





- The **Fibonacci tree** (Fuss-Catalan algebras),
- the **derooted Fibonacci tree** (diagram algebras described by Weber whose 'dual' quantum groups are the freely modified bistochastic quantum groups $B_N^{\#+}$).

diagram algebra	centralizer of	principal graph
Brauer algebras	O_n	Young graph
Partition algebras	S_n	repeated Young graph
rook Brauer algebras	B_n	laddered Young graph
Orellana algebras	H_n	coupled Young graph
walled Brauer algebras	O_n^*	doubled Young graph

Given a tower of diagram algebras

$$A_{(\mathcal{S},\delta)}(0) \subset A_{(\mathcal{S},\delta)}(1) \subset A_{(\mathcal{S},\delta)}(2) \subset \cdots \subset A_{(\mathcal{S},\delta)}(\infty),$$

there is a natural one-to-one correspondence between

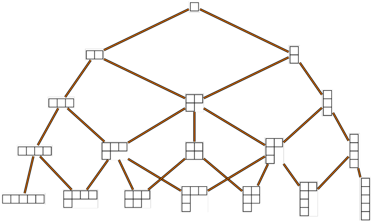
- tracial states τ on the direct limit algebra $A_{(\mathcal{S},\delta)}(\infty)$ and
- measures on the associated branching graph (called **central measures**) satisfying a **consistency condition** that reflects the restriction consistency of traces

$$\left(\tau|_{A_{(\mathcal{S},\delta)}(n+1)}\right)\Big|_{A_{(\mathcal{S},\delta)}(n)} = \tau|_{A_{(\mathcal{S},\delta)}(n)}.$$

- Central measure on branching graph = measure on the space of infinite rooted paths

$$\Omega = \{\emptyset = \omega_0 \nearrow \omega_1 \nearrow \omega_2 \dots\}.$$

- **Consistency**: conditioned on arriving at some $\tilde{\omega}$ at the n-th step, all paths from the root to $\tilde{\omega}$ have been taken with the same probability.



$$\mathbb{P}\left(\ydiagram{1} \nearrow \ydiagram{2} \nearrow \ydiagram{2,1} \middle| \omega_2 = \ydiagram{2,1}\right) = \mathbb{P}\left(\ydiagram{1} \nearrow \ydiagram{1,1} \nearrow \ydiagram{2,1} \middle| \omega_2 = \ydiagram{2,1}\right) = \frac{1}{2}.$$

- The set of central measures forms a **Choquet simplex**, i.e. every central measure can be uniquely represented by a probability measure over its *extreme* points.
- **Problem**:

Compute the **minimal boundary** of the branching graph, i.e. its **extremal central measures**.

- For the Young graph, the classification of extremal central measures is known as **Thoma's theorem**.
- T = set of sequences $((\alpha_n)_{n\geq 1}; (\beta_n)_{n\geq 1}) \in [0,1]^\infty \times [0,1]^\infty$ such that

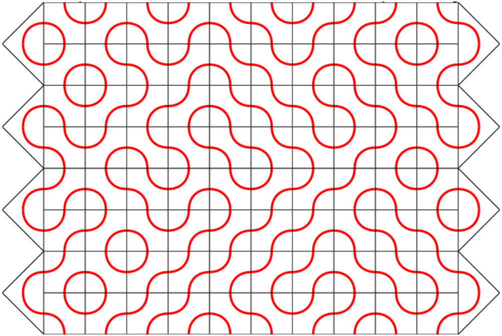
$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0,\ \beta_1 \geq \beta_2 \geq \cdots \geq 0,\ \sum_{n=1}^{\infty}(\alpha_n + \beta_n) \leq 1.$$

diagram algebra	centralizer of	principal graph	boundary principal graph	boundary pascalized graph
Brauer	O_n	Young graph	T	T
Partition	S_n	repeated Young graph	T	T
rook Brauer	B_n	laddered Young graph	T	T
Orellana	H_n	coupled Young graph	T	T?
walled Brauer	O_n^*	doubled Young graph	$T \times T$	$T \times T$

diagram algebra	centralizer of	principal graph	boundary pascalized graph
Temperley-Lieb	O_n^+, S_n^+	$\mathbb{N}$	$[0,1]$
Motzkin	B_n^+	ladder	λ_1, λ_2 s.t. $0 \leq \lambda_2 \leq \lambda_1 \leq 1$, $0 \leq \lambda_1 + \lambda_2 \leq 1$
2-Fuss-Catalan	H_n^+	Fibonacci tree	$[0, 4/27] \times$ $\{$Fibonacci words$\}$*
Weber	$B_n^{\#+}$	derooted Fibonacci tree	$[0, 4/27] \times$ $\{$Fibonacci words$\}$*

*Fibonacci word = word in a, b starting in a s.t. b is always followed by a.

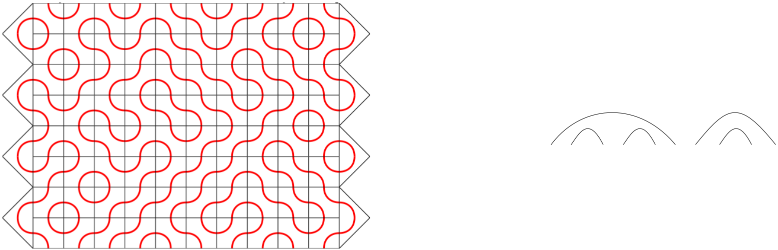
Consider the $O(1)$**-loop model with closed boundary conditions** on a semi-infinite strip of width $2k$:



where every tile is independently drawn with probabilities

$\mathbb{P}($  $) = p$ and $\mathbb{P}($  $) = 1 - p.$

- To every configuration of this model, one can associate a *boundary matching*:



boundary matching of loop model configuration

- Denote by $\mathrm{Ma}(2k)$ the set of matchings (noncrossing pairs on a line);
- Denote by V the space of complex linear combinations of matchings $V_k = \{\sum_{m \in \mathrm{Ma}(2k)} a_m e_m \; ; \; a_m \in \mathbb{C}\}$ with basis $\{e_m, \; m \in \mathrm{Ma}(2k)\}$;

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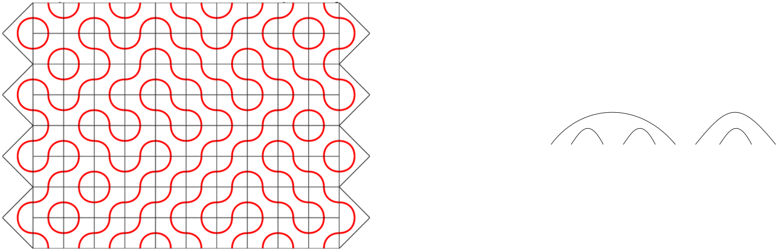

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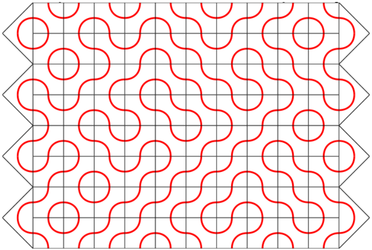

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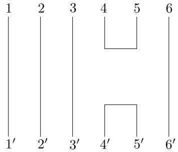
- Then, the Temperley-Lieb algebra $\mathrm{TL}_{\delta=1}(2k)$ (noncrossing pairs) acts on matchings and thus on the space V_k.
- Denote by $\psi(m)$ the probability that the boundary matching in the $O(1)$-loop model is $m \in \mathrm{Ma}(2k)$.
- Consider the vector $\psi_k = \sum_{m \in \mathrm{Ma}(2k)} \psi(m) e_m \in V_k$,
- and the Jones projections $f_i \in \mathrm{TL}_{\delta=1}(2k), i = 1, \ldots 2k-1$.



The Jones projection $f_4 \in \mathrm{TL}_{\delta=1}(6)$.

- Then, ψ_k is invariant under the **Temperley-Lieb Hamiltonian**

$$\left(\sum_{i=1}^{2k-1} f_i\right)\psi_k = \psi_k.$$

- Fascinatingly, all values $\psi_k(m)$ are **integer multiples** of the smallest value.
- The description of these integers is the content of the **Razumov-Stroganov conjecture** for the closed boundary condition.
- This conjecture asserts that these integer enumerates *fully packed loops* and *vertically symmetric alternating sign matrices*.

- For many other boundary conditions (relating to other types of Temperley-Lieb algebras, e.g affine), this conjecture has been proven by Cantini-Sportiello '10.
- There are also loop models for the Brauer, the Motzkin and the Fuss-Catalan algebra.
- For the **Brauer algebra**, there is a similar integer multiplicity phenomenon, relating the model to the degree of certain algebraic varieties (Nienhuis '04, Knutson and Zinn-Justin '05).

Thanks for

listening!